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Comput. Methods Appl. Mech. Engrg. 195 (2006) 2574–2598

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

Reliable and efficient equilibrated a posteriori finite element error control in elastoplasticity and elastoviscoplasticity with hardening

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Received 5 July 2004; received in revised form 25 May 2005; accepted 27 May 2005

Abstract

This paper establishes the reliability and efficiency of equilibrated a posteriori estimates for L^2 stress error control of conforming displacement finite element approximations of incremental plasticity and viscoplasticity with hardening. Explicit expressions for upper bounds of the reliability constant that enters the guaranteed upper error bound illustrate its crucial dependence on the hardening material constants. Numerical experiments show that adaptive finite element solutions with marking strategy based on the max-refinement rule and local equilibrated error estimators lead to optimal empirical convergence rates.

Keywords: Elastoviscoplasticity; Hardening; Finite element method; A posteriori error estimate; Reliability; Efficiency; Adaptive algorithms

1. Introduction

Within each time step in a finite element analysis of elastoplastic and elastoviscoplastic evolution problems, one must solve a variational inequality with a complicated material law determined by admissible (generalised) stresses on top of the problem of linear elasticity. In the development of a posteriori error estimates for this class of problems one would expect that the estimates should involve also a measure of the residual in the material law, e.g., in some Kuhn–Tucker conditions on the plastic multiplier. However, as

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0045-7825/$ - see front matter
doi:10.1016/j.cma.2005.05.016

reported in [25], estimates of the norm of the residual in the equilibrium conditions might serve as an error estimator for reliable error control of finite element approximations of incremental plasticity and viscoplasticity. Corresponding residual-based estimates contain the term

$$\eta_{T,\mathrm{R}} = h_T^2 \int_T |f + \mathrm{div}_{\mathscr{T}}\, \sigma_h|^2 \mathrm{d}x + \int_{\partial T} h_E |[\sigma_h \nu_E]|^2 \,\mathrm{d}s, \tag{1.1}$$

for one element T (of diameter h_T) with edges E (of length h_E) on its boundary ∂T; f is a given volume force and $\mathrm{div}_{\mathscr{T}}\sigma_h$ is the piecewise divergence while $[\sigma_h \nu_E]$ denotes the jump of the normal tractions $\sigma_h \nu_E$ across the element edge E in the direction ν_E (and standard modification on parts of the boundary of Ω with applied surface loads). The estimator of [25] presents, however, also other terms related to the plastic region where the functional analytical setting required for perfect plasticity provides only very weak approximation properties of the displacement field in $\mathrm{BD}(\Omega)$ [40,20]. It is shown in [4,11] that (1.1) yields a reliable and efficient error estimator

$$\eta_{\mathrm{R}} = \left(\sum_{T \in \mathscr{T}} \eta_T^2 \right)^{1/2}, \tag{1.2}$$

that is, there exist constants $C_{\mathrm{R,eff}}$ and $C_{\mathrm{R,rel}}$ and higher order terms (hot) such that the following bounds hold:

$$C_{\mathrm{R,eff}}\eta_{\mathrm{R}} \leqslant \|\sigma - \sigma_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \mathrm{hot}, \quad \|\sigma - \sigma_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leqslant C_{\mathrm{R,rel}}\eta_{\mathrm{R}} + \mathrm{hot}. \tag{1.3}$$

The estimates $(1.3)_1$ and $(1.3)_2$ are next referred to as efficiency and reliability estimate, respectively. The constants $C_{\mathrm{R,eff}}$ and $C_{\mathrm{R,rel}}$ depend neither on the mesh-sizes h_T, h_E nor on the unknown exact solution [14]; they depend on the domain, the material law and material (hardening and viscosity) parameters. For more general error norms (or error functionals) the duality approach of [37,38,33,17] could be employed. The residuals of the computed solution are multiplied by local weights obtained numerically from the solution of linearized dual problems. Although the theoretical justification of these estimates is disputable, this approach leads to very accurate error guesses and is very valuable for particular error functionals.

In this paper, we consider L^2 stress error control in elastoplasticity and viscoplasticity with hardening [11,4,12,15]. Therein, in each time step, the functional analytical context of linear elasticity is applicable on the price of that some constants crucially depend on the hardening moduli. Averaging [12], explicit residual type [25,11,4], and heuristic estimates [8,36] of the error have been considered and analysed for the finite element error control of incremental plasticity. In these contributions, the above dependence has been always mentioned but never investigated. For explicit residual-based a posteriori error estimators it is moreover known for elliptic problems [13] that the strict estimation of $C_{\mathrm{R,rel}}$ often leads to a huge overestimation and hence appears almost useless as stopping criterion. For sharp error control, on contrary, the implicit error estimators such as the equilibration error estimator [28,2] have proved to give a better performance [13]. This is also confirmed by several applications of the equilibration techniques for the construction of admissible solutions used in the family of error measures based on the error in the constitutive equations for associative material models [21,30].

The main goals of this work are therefore the following. First, we want to explore in an explicit way the type of dependence of the reliability constants on the material properties, in particular on the hardening modulus and the viscosity coefficient for a quite general class of inelastic material models. Instrumental are the establishment of some bounds on the plastic strain error in terms of the stress error, useful also in a broader context than error control of finite element approximations. Second, we want to analyse the feasibility of equilibration technique for efficient and reliable error control of the space discretization error. Finally, we want to investigate the numerical performance of adaptive finite element schemes with marking strategy based on the max-refinement rule and on the local equilibrated error estimators $\eta_{\mathrm{EQ},T}$.

Likewise in linear elasticity, the local error indicators $\eta_{\mathrm{EQ},T}$ are related to the Riesz representation in a suitable Hilbert space $\mathscr{W}_T$ to be made precise later of the local residual ℓ_T^{EQ},

$$\ell_T^{\mathrm{EQ}} := \int_T r_{h,T} \cdot v \,\mathrm{d}x + \sum_{E\subseteq\partial T} \int_E J_{h,E}^T \cdot v \,\mathrm{d}s, \tag{1.4}$$

where $J_{h,E}^T$ with $E \subseteq \partial T$ arises from the equilibration splitting of the residual jump $J_{h,E}$ resulting by the lack of equilibrium of the normal tractions $\sigma_h \nu_E$ across E and $r_{h,T}$ is associated with the lack of equilibrium at the interior of each element $T \in \mathscr{T}$. The incremental form of the evolution equations describing the plastic material law are satisfied *exactly* at the Gauss points of each element, hence the material law has therein a vanishing residual. Excluding the error accumulation for progressing time steps, the only remaining residuals are, therefore, the discrete equilibrium conditions of (1.1). A posteriori error analysis of the time discretization error is, however, not straightforward to involve, hence in this paper we focus only on one time increment. For an analysis based on first principles of the all important effects of time discretization one can, however, refer to [5,21,31,32,35].

The remaining part of the paper is organized as follows. Section 2 introduces notation and setting of the problem under consideration, Section 3 presents a general class of constitutive models in elastoplasticity and elastoviscoplasticity. This prepares the ground for the main results of the paper in Section 4. Here, the reliability and efficiency of equilibrated residual type error estimates for the L^2 norm of the error on the stress of conforming displacement finite element approximation of the incremental elastoplastic and elastoviscoplastic problem are stated and proved. It turns out that the reliability constant $C_{\mathrm{R,rel}}$ depends crucially on the hardening material constants whereas the efficiency constant $C_{\mathrm{R,eff}}$ depends only on the regularity of the triangulation. The type of dependence of $C_{\mathrm{R,rel}}$ is established for the class of inelastic material models introduced in Section 3 and it is shown that also in presence of viscosity the estimate degenerates for values of the hardening moduli small compared to the elastic modulus. Section 5 finally reports on the behaviour of the estimates in terms of the hardening parameter and on adaptive strategies based on the local error estimators $\eta_{\mathrm{EQ},T}$ and (1.1). Numerical experiments on model problems show that if we rely on the efficiency estimate, the adaptive refinement process performs improved convergence rates.

2. Setting of the problem

This section introduces the displacement formulation for stress-based elastoplastic and viscoplastic constitutive models with internal variables, more frequently found in the engineering literature.

2.1. Constitutive model and continuous formulation

Let $(0, T)$ denote the time interval of interest, Ω a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 2, 3, \ldots$ with boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, and ν the outer unit normal. The boundary $\partial\Omega$ is split into a closed Dirichlet part Γ_{D} with positive surface measure and into a Neumann part $\Gamma_{\mathrm{N}} := \partial\Omega \backslash \Gamma_{\mathrm{D}}$ (possibly empty) where traction forces are prescribed by $g \in L^2(\Gamma_{\mathrm{N}};\ \mathbb{R}^d)$. The strong form of equilibrium conditions states that the stress field $\sigma \in \mathscr{S} := L^2(\Omega;\ \mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ being the set of all real symmetric $d \times d$ matrices, satisfies

$$\operatorname{div} \sigma + f = 0 \text{ in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma \nu = g \text{ on } \Gamma_{\mathrm{N}} \times (0, T), \tag{2.2}$$

where $f \in L^2(\Omega; \mathbb{R}^d)$ are applied volume force. We suppose, for sake of simplicity, homogeneous geometric boundary conditions for the displacement field u and introduce the following space of admissible displacements:

$$V := \{v \in H^1(\Omega; \mathbb{R}^d) : \ v = 0 \text{ on } \Gamma_{\mathrm{D}}\},$$

equipped with the $H^1(\Omega; \mathbb{R}^d)$-norm and boundary values to be interpreted in the sense of the trace theorem; V is a closed subspace of $H^1(\Omega; \mathbb{R}^d)$.

The data f and g are used to define, thanks to the trace theorem, the bounded functional $\ell := \langle f, v\rangle + \langle g, v\rangle_{\Gamma_{\mathrm{N}}} \in V^*$, V^* being the topological dual of V, and $\langle\cdot,\cdot\rangle$ the duality pairing V and V^*.

Given the fourth-order isotropic elasticity tensor $\mathbb{C}$, such that $\mathbb{C}\delta = 2\mu\delta + \lambda \operatorname{tr}\delta I$ for $\delta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, we suppose an additive split $\varepsilon(u) = \mathbb{C}^{-1}\sigma + p$ of the (linear) Green strain

$$\varepsilon(u) = \operatorname{sym}(Du) = ((u_{j,k} + u_{k,j})/2 : j, k = 1, \ldots, d),$$

into an elastic $e = \mathbb{C}^{-1}\sigma$ and plastic part p. After introducing further internal (hardening) variables $\alpha \in \mathscr{M} := L^2(\Omega; \mathbb{R}^m)$ and the hardening material tensor $\mathbb{H} \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$ we consider an associative material law

$$(\varepsilon(\dot{u}) - \mathbb{C}^{-1}\dot{\sigma}) : (\tau - \sigma) - \mathbb{H}^{-1}\dot{\alpha} \cdot (\beta - \alpha) \leqslant j(\tau, \beta) - j(\sigma, \alpha), \tag{2.3}$$

for all $(\tau, \beta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m$. We assume the plastic potential $j : \ \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \to [0, \infty]$ be convex, lower semicontinuous, proper (i.e., $j \not\equiv +\infty$), and that hardening parameters guarantee positive definiteness of the bilinear form.

Here, in (2.3), and below, the symbol ':' denotes the scalar product of two second order tensors, i.e. $\sigma : \varepsilon = \sum_{i,j=1}^{d} \sigma_{ij}\varepsilon_{ij}$, the symbol '·' is the scalar product in the Euclidean space $\mathbb{R}^m$, and '$|\cdot|$' is the associated norm. The Lebesgue and Sobolev vector spaces are defined in the usual way.

2.2. Time discretization

A generalised midpoint rule serves as a time discretization. Each time step results in a spatial problem with given variables $(u(t), \sigma(t), \alpha(t))$ at time t_0 denoted as $(u_0, \sigma_0, \alpha_0)$ and unknowns at the time $t_1 = t_0 + \Delta t$ denoted as $(u_1, \sigma_1, \alpha_1)$. Time derivatives are replaced by backward differences quotients. Let $(\cdot)_\theta = \theta(\cdot)_1 + (1-\theta)(\cdot)_0$, with $1/2 \leqslant \theta \leqslant 1$ for the stability of the numerical scheme [23].

Problem 2.1 (*The time discrete problem*). Given $\ell_\theta \in V^*$ and $(u_0, \sigma_0, \alpha_0) \in V \times \mathscr{S} \times \mathscr{M}$, seek $u_\theta \in V$ such that

$$\begin{aligned}
&\int_\Omega \sigma_\theta : \varepsilon(v)\,\mathrm{d}x = \ell_\theta(v) \quad \text{for all } v \in V \text{ and} \\
&\int_\Omega ((\varepsilon(u_\theta - u_0) - \mathbb{C}^{-1}(\sigma_\theta - \sigma_0)) : (\tau - \sigma_\theta) - \mathbb{H}^{-1}(\alpha_\theta - \alpha_0)\cdot(\beta - \alpha_\theta))\mathrm{d}x \\
&\quad \leqslant \theta\Delta t \int_\Omega j(\tau, \beta)\mathrm{d}x - \theta\Delta t \int_\Omega j(\sigma_\theta, \alpha_\theta)\mathrm{d}x \quad \text{for all } (\tau, \beta) \in \mathscr{S} \times \mathscr{M}.
\end{aligned} \tag{2.4}$$

The displacement formulation of the stress formulation requires to solve pointwise the following incremental constitutive law.

Problem 2.2 (*The incremental constitutive law*). Given the initial data $(\varepsilon_0, \sigma_0, \alpha_0) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m$ and the total strain $\varepsilon_\theta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, seek $(\sigma_\theta, \alpha_\theta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m$ such that

$$\begin{aligned}
&((\varepsilon_\theta - \varepsilon_0) - \mathbb{C}^{-1}(\sigma_\theta - \sigma_0)) : (\tau - \sigma_\theta) - \mathbb{H}^{-1}(\alpha_\theta - \alpha_0)\cdot(\beta - \alpha_\theta) \\
&\quad \leqslant \theta\Delta t j(\tau, \beta) - \theta\Delta t j(\sigma_\theta, \alpha_\theta) \quad \text{for all } (\tau, \beta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m.
\end{aligned} \tag{2.5}$$

By rearranging the terms in (2.5) as follows:

$$\begin{aligned}(\varepsilon_\theta - (\varepsilon_0 - \mathbb{C}^{-1}\sigma_0)) : (\tau - \sigma_\theta) + \mathbb{H}^{-1}\alpha_0 \cdot (\beta - \alpha_\theta) \\ \leqslant \theta\Delta t j(\tau, \beta) - \theta\Delta t j(\sigma_\theta, \alpha_\theta) + \mathbb{C}^{-1}\sigma_\theta : (\tau - \sigma_\theta) + \mathbb{H}^{-1}\alpha_\theta \cdot (\beta - \alpha_\theta),\end{aligned} \tag{2.6}$$

Problem 2.2 defines an elliptic variational inequality, usually referred to as of second kind, which has solution and is unique [22].

Remark 2.1. Inequality (2.5) shows that for given ε_0, σ_0, α_0, the operator $\varepsilon_\theta \mapsto \sigma_\theta$ is monotone and Lipschitz continuous, that is, there exists a positive constant dependent on the elasticity tensor such that the following inequalities hold

$$c|\sigma_1 - \sigma_2|^2 \leqslant (\sigma_1 - \sigma_2) : (\varepsilon_1 - \varepsilon_2) \text{ and } |\sigma_1 - \sigma_2| \leqslant c|\varepsilon_1 - \varepsilon_2|, \tag{2.7}$$

for solutions (σ_1, α_1) and (σ_2, α_2) of (2.5) and $\varepsilon_\theta = \varepsilon_1$ and $\varepsilon_\theta = \varepsilon_2$, respectively.

Proof. Since (σ_j, α_j) for $j = 1, 2$ are solutions of (2.5) one obtains

$$((\varepsilon_1 - \varepsilon_0) - \mathbb{C}^{-1}(\sigma_1 - \sigma_0)) : (\sigma_2 - \sigma_1) - \mathbb{H}^{-1}(\alpha_1 - \alpha_0) \cdot (\alpha_2 - \alpha_1) \leqslant \theta\Delta t j(\sigma_2, \alpha_2) - \theta\Delta t j(\sigma_1, \alpha_1), \tag{2.8}$$

$$((\varepsilon_2 - \varepsilon_0) - \mathbb{C}^{-1}(\sigma_2 - \sigma_0)) : (\sigma_1 - \sigma_2) - \mathbb{H}^{-1}(\alpha_2 - \alpha_0) \cdot (\alpha_1 - \alpha_2) \leqslant \theta\Delta t j(\sigma_1, \alpha_1) - \theta\Delta t j(\sigma_2, \alpha_2). \tag{2.9}$$

The sum of Eqs. (2.8) and (2.9) yields

$$\mathbb{C}^{-1}(\sigma_1 - \sigma_2) : (\sigma_1 - \sigma_2) + \mathbb{H}^{-1}(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2) \leqslant (\sigma_1 - \sigma_2) : (\varepsilon_1 - \varepsilon_2). \tag{2.10}$$

The positive definiteness of $\mathbb{C}$ leads to

$$\frac{1}{d\lambda + \mu}|\sigma_1 - \sigma_2|^2 \leqslant \mathbb{C}^{-1}(\sigma_1 - \sigma_2) : (\sigma_1 - \sigma_2). \tag{2.11}$$

The combination of Eqs. (2.10) and (2.11) with the positive definiteness of $\mathbb{H}$ proves the first inequality in (2.7); the second follows from the first and a Cauchy inequality. $\square$

2.3. Space discretization

The spatial discrete problem involves a shape-regular triangulation $\mathcal{T}$ (in triangles if $d = 2$ or tetrahedrons if $d = 3$ etc.) of the domain Ω in the sense of Ciarlet [16,10]. Hanging nodes are excluded and $\cup\mathcal{T}$ matches exactly the domain and its boundary,

$$\cup\mathcal{T} = \overline{\Omega} = \Omega \cup \partial\Omega.$$

Here and in the sequel, $\mathcal{E}$ is the set of all edges in $\mathcal{T}$. For each element domain $T \in \mathcal{T}$ of diameter $h_T := \mathrm{diam}(T)$ and area $|T| > 0$, $\mathcal{E}(T) \subset \mathcal{E}$ denotes the set of the edges of T. Furthermore, $\omega(T)$ denotes the patch of elements T' sharing a common edge with T, $\omega(E)$ the patch of elements having edge E, whereas $\mathring{\mathcal{E}}(\omega(T))$ denotes the interior edges of the patch $\omega(T)$. Each edge E with $E \subset \partial\Omega$ satisfies either $E \subset \Gamma_{\mathrm{D}}$ or $E \subset \overline{\Gamma}_{\mathrm{N}}$ and this is written as $E \in \mathcal{E}_{\mathrm{D}}$ and $E \in \mathcal{E}_{\mathrm{N}}$, respectively. All the remaining edges satisfy $E \not\subset \partial\Omega$ and are written as $E \in \mathcal{E}_\Omega$. This defines a partition

$$\mathcal{E} = \mathcal{E}_\Omega \cup \mathcal{E}_{\mathrm{D}} \cup \mathcal{E}_{\mathrm{N}}.$$

Given any edge $E \in \mathcal{E}$ of length $h_E := \mathrm{diam}(E)$ there is one fixed unit normal ν_E; for $E \in \mathcal{E}_{\mathrm{D}} \cup \mathcal{E}_{\mathrm{N}}$ on the boundary we choose $\nu_E = \nu$. Once ν_E has been fixed on E, in relation to ν_E one defines the elements $T_+ \in \mathcal{T}$ and $T_- \in \mathcal{T}$, with $E \subseteq \partial T_- \cap \partial T_+$, as depicted in Fig. 1.

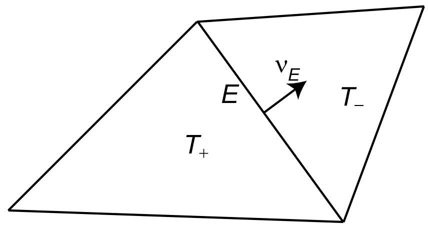

Fig. 1. Definition of the elements T_+ and T_- in relation to ν_E.

Furthermore, given $E \in \mathscr{E}_\Omega$ with unit normal ν_E and v a vector field defined in Ω, we denote by $[v]_E$ the jump of v across E in the direction ν_E, i.e.

$$[v]_{\mathrm{E}}(x) = v|_{T_+}(x) - v|_{T_-}(x) \quad \text{for } x \in E \text{ and } E \subseteq \partial T_- \cap \partial T_+.$$

The finite element method is defined by the finite element space V_h of the trial and test function space, here coincident. For any subset ω (patch, triangle, or edge), let $P_k(\omega; \mathbb{R}^d)$ denote the vector space of all algebraic polynomials in d variables on ω of total degree at most $k \in \mathbb{N}$. Then,

$$P_k(\mathscr{T}; \mathbb{R}^d) := \{v_h \in L^\infty(\Omega; \mathbb{R}^d) : \ \forall T \in \mathscr{T}, v_h|_T \in P_k(T; \mathbb{R}^d)\}$$

denotes the piecewise polynomials of degree at most k where piecewise is with respect to the shape-regular triangulation $\mathscr{T}$; in general, the functions in $P_k(\mathscr{T}; \mathbb{R}^d)$ are discontinuous. The globally continuous functions in $P_k(\mathscr{T}; \mathbb{R}^d)$ form the P_k finite element spaces,

$$P_k = P_k(\mathscr{T}; \mathbb{R}^d) \cap C(\overline{\Omega}; \mathbb{R}^d) \quad \text{ and } V_h := P_k(\mathscr{T}; \mathbb{R}^d) \cap V.$$

The conforming finite element approximation of Problem 2.1 reads as follows.

Problem 2.3 (*Fully discrete problem*). Given $\ell_\theta \in V^*$ and $(u_{0,h}, \sigma_{0,h}, \alpha_{0,h}) \in V_h \times \mathscr{S} \times \mathscr{M}$, seek $u_{\theta,h} \in V_h$ such that

$$\int_\Omega \sigma_{\theta,h} : \varepsilon(v)\mathrm{d}x = \ell_\theta(v) \quad \text{for all } v \in V_h \text{ and} \tag{2.12}$$

$$\begin{aligned}&\int_\Omega ((\varepsilon(u_{\theta,h} - u_{0,h}) - \mathbb{C}^{-1}(\sigma_{\theta,h} - \sigma_{0,h})) : (\tau - \sigma_{\theta,h}) - \mathbb{H}^{-1}(\alpha_{\theta,h} - \alpha_{0,h}) \cdot (\beta - \alpha_{\theta,h}))\mathrm{d}x \\ &\quad \leqslant \theta\Delta t \int_\Omega j(\tau, \beta)\mathrm{d}x - \theta\Delta t \int_\Omega j(\sigma_{\theta,h}, \alpha_{\theta,h})\mathrm{d}x \quad \text{for all } (\tau, \beta) \in \mathscr{S} \times \mathscr{M}.\end{aligned} \tag{2.13}$$

Remark 2.2. Problem 2.3 involves a finite element approximation exclusively for the displacement. The variables (σ_h, α_h) are obtained by the pointwise solution of the incremental constitutive Problem 2.2.

We refer to [24] for details on existence and uniqueness of discrete solutions established for P_1 triangular finite elements, and mention only that (2.12) is the discrete weak form of Eqs. (2.1) and (2.2) whereas (2.13) is the equivalent integral form of (2.5).

3. Examples in elastoplasticity with hardening and viscoplastic regularization

This section introduces a general class of inelastic material models that will be analysed in this paper.

3.1. Plasticity with combined isotropic/kinematic hardening

Let $m = 1 + d(d+1)/2$, identify $\mathbb{R}^m \equiv \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}$, and write $\alpha = (\chi, a)$ where χ is the back stress tensor and a is the accumulated plastic strain which is a nondecreasing and nonnegative function of the time. Introduce the elastic domain

$$\mathbb{E} := \{(\sigma,\chi,a) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R} | \phi(\sigma,\chi,a) \leqslant 0\}, \tag{3.1}$$

where $\phi(\sigma,\chi,a)$ is the continuous and convex von-Mises yield function

$$\Phi(\sigma,\chi,a) := |\operatorname{dev}\sigma - \operatorname{dev}\chi| - (\sigma_{\text{y}} + Ha), \tag{3.2}$$

in case $a \geqslant 0$ (and $\Phi(\sigma,\alpha) = \infty$ if $a < 0$ which, thereby, is not allowed). In (3.2) the material constant $\sigma_{\text{y}} > 0$ is the yield stress and the constant $H > 0$ is the hardening modulus. If ξ is the kinematic-type internal variable conjugate of χ, we assume $\chi = k\xi$ which is the Melan–Prager law and $k > 0$ a material constant.

The evolution law of the kinematic-type internal variables (p, ξ, a) are obtained by choosing $j = I_{\mathbb{E}}$ in (2.3), with $I_{\mathbb{E}}$ indicator function of $\mathbb{E}$ ([19]). As a result, (2.3) is recast as

$$(\dot{p}, -\dot{\xi}, -\dot{a}) \in \partial I_{\mathbb{E}}. \tag{3.3}$$

3.2. Viscoplastic regularization

Following [18] the viscoplastic material model is defined as a Yosida regularization of the plasticity material model. If $\Pi_{\mathbb{E}}$ denotes the L^2-projection of the generalised stress $\Sigma = (\sigma, \chi, Ha) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}$ onto the elastic domain $\mathbb{E}$ and ϱ is the regularization parameter, which can be considered as a viscosity coefficient, the evolution of the internal variables are defined as follows.

Let

$$j(\Sigma) := \frac{1}{\varrho}|\Sigma - \Pi_{\mathbb{E}}\Sigma|^2 \tag{3.4}$$

from (2.3) one obtains [23]

$$(\dot{p}, -\dot{\xi}, -\dot{a}) = \frac{\partial j(\Sigma)}{\partial \Sigma} = \frac{1}{\varrho}(\Sigma - \Pi_{\mathbb{E}}\Sigma). \tag{3.5}$$

By letting $\varrho \to 0$, the model (3.3) of rate-independent plasticity is retrieved [18]. For elastoviscoplasticity with linear kinematic/isotropic hardening and von Mises yield function, once $\Pi_{\mathbb{E}}$ is evaluated, from (3.5) it follows:

$$\begin{aligned}
\dot{p} &= \frac{1}{3\varrho}(|\operatorname{dev}(\sigma - k\xi)| - (\sigma_{\text{y}} + aH))_+ \frac{\operatorname{dev}(\sigma - k\xi)}{|\operatorname{dev}(\sigma - k\xi)|},\\
\dot{\xi} &= \frac{1}{3\varrho}(|\operatorname{dev}(\sigma - k\xi)| - (\sigma_{\text{y}} + aH))_+ \frac{\operatorname{dev}(\sigma - k\xi)}{|\operatorname{dev}(\sigma - k\xi)|},\\
\dot{a} &= \frac{1}{3\varrho}(|\operatorname{dev}(\sigma - k\xi)| - (\sigma_{\text{y}} + aH))_+,
\end{aligned} \tag{3.6}$$

where for $a \in \mathbb{R}$, $(a)_+ = \max\{0, a\}$. Hence, the incremental evolution law (2.5) reads

$$\begin{aligned}
p_\theta - p_0 &= \frac{\theta\Delta t}{3\varrho}(|\operatorname{dev}(\sigma_\theta - k\xi_\theta)| - (\sigma_{\text{y}} + a_\theta H))_+ \frac{\operatorname{dev}(\sigma_\theta - k\xi_\theta)}{|\operatorname{dev}(\sigma_\theta - k\xi_\theta)|},\\
\xi_\theta - \xi_0 &= \frac{\theta\Delta t}{3\varrho}(|\operatorname{dev}(\sigma_\theta - k\xi_\theta)| - (\sigma_{\text{y}} + a_\theta H))_+ \frac{\operatorname{dev}(\sigma_\theta - k\xi_\theta)}{|\operatorname{dev}(\sigma_\theta - k\xi_\theta)|},\\
a_\theta - a_0 &= \frac{\theta\Delta t}{3\varrho}(|\operatorname{dev}(\sigma_\theta - k\xi_\theta)| - (\sigma_{\text{y}} + Ha_\theta))_+.
\end{aligned} \tag{3.7}$$

Remark 3.1. For the model (3.7), Problem 2.2 can be solved in closed form with respect to the trial elastic state $(\varepsilon_\theta - (\varepsilon_0 - \mathbb{C}^{-1}\sigma_0), \alpha_0)$ [39,15].

4. Reliability and efficiency of η_{EQ}

In this section after recalling the equilibration technique, we state and prove the main results of the paper on the reliability and efficiency of equilibrated residual type error estimates for the L^2 norm of the error on the stress.

4.1. Equilibrated residual error estimates

Let σ_θ and $\sigma_{\theta,h}$ be solution of Problems 2.1 and 2.3, respectively. The residual functional produced by the stress field $\sigma_{\theta,h}$ in the equilibrium equations is the element of V^* defined by

$$\langle R_h, v\rangle := \ell_\theta(v) - \int_\Omega \sigma_{\theta,h} : \varepsilon(v)\mathrm{d}x = \int_\Omega (\sigma_\theta - \sigma_{\theta,h}) : \varepsilon(v)\mathrm{d}x. \tag{4.1}$$

Equilibrated error estimates are based on

(i) a localization of the global residual $\langle R_h, v\rangle$ into equilibrated element residuals,

$$\langle R_h, v\rangle = \sum_{T\in\mathscr{T}} \ell_T^{\mathrm{EQ}}(v) \quad \text{for all } v \in V \tag{4.2}$$

with the property that

$$\ell_T^{\mathrm{EQ}}(v) = 0 \quad \text{for all } v \in \mathrm{RM}(T). \tag{4.3}$$

(ii) a Riesz representation of the bounded linear functional ℓ_T^{EQ} in the Hilbert space $\mathscr{W}_T$.

If $T \in \mathscr{T}$ is such that $\partial T \cap \Gamma_{\mathrm{D}} \neq \emptyset$ and $\partial T \cap \Gamma_{\mathrm{D}} \subseteq \mathscr{E}_{\mathrm{D}}$, the local space $\mathscr{W}_T$ is defined as

$$\mathscr{W}_T := H^1_{\mathrm{D}}(T;\mathbb{R}^d) := \{w \in H^1(T;\mathbb{R}^d) : w = 0 \text{ on } \partial T \cap \Gamma_{\mathrm{D}}\}, \tag{4.4}$$

otherwise we assume

$$\mathscr{W}_T := H^1(T;\mathbb{R}^d)/\mathrm{RM}(T). \tag{4.5}$$

Here, $\mathrm{RM}(T)$ denotes the kernel of $\varepsilon(v|_T)$, namely the set of the rigid body motions on the element $T \in \mathscr{T}$ [34]. In the latter case, $(\int_T \varepsilon(v) : \varepsilon(v)\mathrm{d}x)^{1/2}$ is a norm in $\mathscr{W}_T$, for it is equivalent to the quotient norm thanks to the Korn inequality and the equivalence of the seminorm $\|Du\|_{L^2(T;\mathbb{R}^{d\times d})}$ to the quotient norm in (4.5) [16,34].

The localization of the integrals in (4.1) over each $T \in \mathscr{T}$, use of the integration by parts, and suitable rearrangement of the several terms on the boundaries yields [6]

$$\langle R_h, v\rangle = \sum_{T\in\mathscr{T}} \int_T r_{h,T} \cdot v\,\mathrm{d}x + \sum_{E\in\mathscr{E}} \int_E J_{h,E} \cdot v\,\mathrm{d}s. \tag{4.6}$$

In (4.6), $r_{h,T} := \mathrm{div}_{\mathscr{T}}\sigma_{\theta,h} + f$ defines the regular component of $\langle R_h, v\rangle$ associated with the lack of equilibrium at the interior of each element $T \in \mathscr{T}$, whereas for each edge $E \in \mathscr{E}, J_{h,E}$ defines the singular component and denotes the jump of the normal tractions of the stress fields across E in direction ν_E,

$$J_{h,E} = \begin{cases} [\sigma_h \nu_E]_{\mathrm{E}} & \text{if } E \in \mathscr{E}_\Omega, \\ g - \sigma_h \nu & \text{if } E \in \mathscr{E}_{\mathrm{N}}, \\ 0 & \text{if } E \in \mathscr{E}_{\mathrm{D}}. \end{cases} \tag{4.7}$$

Notice that $[\sigma_h \nu_E]_E$ does not depend on the orientation of ν_E.

Localization of the residual into equilibrated element residuals is obtained by assuming a splitting of the interelement flux $J_{h,E}$ meeting the following conditions:

$$\begin{aligned} &J_{h,E}^{T_+} + J_{h,E}^{T_-} = J_{h,E}, \\ &\ell_T^{\mathrm{EQ}}(v) := \int_T r_{h,T} \cdot v \,\mathrm{d}x + \sum_{E \in \mathscr{E}(T)} \int_E J_{h,E}^T \cdot v \,\mathrm{d}s = 0 \quad \text{for all } v \in P_k(T;\mathbb{R}^d), \end{aligned} \tag{4.8}$$

with $J_{h,E}^T = J_{h,E}$ if $E \in \mathscr{E}_{\mathrm{N}} \cup \mathscr{E}_{\mathrm{D}}$ [6, p. 500]. Here, $P_k(T;\mathbb{R}^d)$ denotes the local finite element space and $(4.8)_2$ is referred to as pth-order equilibration or prolongation condition [2,28]. Since $\mathrm{RM}(T) \subseteq P_k(T;\mathbb{R}^d)$, condition (4.3) is enforced as well. Hence, $\langle R_h, v\rangle$ reads

$$\langle R_h, v\rangle = \sum_{T\in\mathscr{T}} \ell_T^{\mathrm{EQ}}(v) = \sum_{T\in\mathscr{T}} \int_T r_{h,T} \cdot v \,\mathrm{d}x + \sum_{E\in\mathscr{E}(T)} \int_E J_{h,E}^T \cdot v \,\mathrm{d}s. \tag{4.9}$$

For the definition of ℓ_T^{EQ}, several equilibration techniques have been proposed [27,28,7,3,29] according to the definition of the splitting of the interelement flux $J_{h,E}$ and the solution of the resulting system in terms of the degrees of freedom that define the polynomial pointwise distribution of the element boundary tractions $J_{h,E}^T$ on $E \in \mathscr{E}(T)$. In this paper, we consider the flux splitting given in [3], that is, for $E \in \mathscr{E}_\Omega$, let

$$J_{h,E}^{T_+} = g_E|_{T_+} - \sigma_h|_{T_+} \nu_{T_+} \quad \text{and } J_{h,E}^{T_-} = g_E|_{T_-} - \sigma_h|_{T_-} \nu_{T_-}, \tag{4.10}$$

with $g_E|_{T_+} + g_E|_{T_-} = 0, \nu_T$ the outward normal to T, and $\sigma_h|_T$ the restriction of σ_h to T, whereas for $E \in \mathscr{E}_{\mathrm{N}} \cup \mathscr{E}_{\mathrm{D}}, J_{h,E}^T = J_{h,E}$.

The Riesz representation of the bounded functionals $\ell_T^{\mathrm{EQ}}(v)$ is obtained by the following theorem.

Theorem 4.1. *Given any $T \in \mathscr{T}$, there exists a unique $\phi_T \in \mathscr{W}_T$ with*

$$\int_T \varepsilon(\phi_T) : \varepsilon(v)\,\mathrm{d}x = \ell_T^{\mathrm{EQ}}(v) \quad \textit{for all } v \in \mathscr{W}_T. \tag{4.11}$$

Proof. If $W_T = H^1_{\mathrm{D}}(T;\mathbb{R}^d)$, using the second Korn inequality, one can easily show that Lax–Milgram theorem applies. If $W_T = H^1(T;\mathbb{R}^d)/\mathrm{RM}(T)$, since $\ell_T^{\mathrm{EQ}}(v) = 0$ for all $v \in \mathrm{RM}(T)$, the functional $\ell_T^{\mathrm{EQ}}(v)$ is well defined over $H^1(T;\mathbb{R}^d)/\mathrm{RM}(T)$ and is therein continuous. Hence, the Riesz theorem applies and proves the assertion. □

Definition 4.1 (*Error estimator η_{EQ}*). Given the finite element stress field σ_h, and an exact splitting of the residual jump $J_{h,E}$ into $J_{h,E}^{T_+}$ and $J_{h,E}^{T_-}$ (and standard modification if $E \subseteq \Gamma_{\mathrm{N}}$) meeting conditions (4.8), define the functional $\ell_T^{\mathrm{EQ}}(v)$ for $v \in \mathscr{W}_T$ by

$$\ell_T^{\mathrm{EQ}}(v) := \int_T r_{h,T} \cdot v \,\mathrm{d}x + \sum_{E\in\mathscr{E}(T)} \int_E J_{h,E}^{T_+} \cdot v \,\mathrm{d}s.$$

For each element $T \in \mathscr{T}$ define $\eta^2_{\mathrm{EQ},T}$ as

$$\eta^2_{\mathrm{EQ},T} := \|\varepsilon(\phi_T)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} = \int_T \varepsilon(\phi_T) : \varepsilon(\phi_T)\mathrm{d}x. \tag{4.12}$$

As a global error estimator define

$$\eta_{\mathrm{EQ}} := \left(\sum_{T\in\mathscr{T}} \eta^2_{\mathrm{EQ},T}\right)^{1/2}. \tag{4.13}$$

4.2. Main results

The equilibration technique described in the previous section provides reliable and efficient estimates for the L^2 norm of the error on the stress of conforming displacement finite element approximations of the incremental Problem 2.1.

Theorem 4.2. *Assume Problems* 2.1 *and* 2.3 *are defined by the same initial data, i.e.* $u_0 = u_{0,h}$, $\sigma_0 = \sigma_{0,h}$, *and* $\alpha_0 = \alpha_{0,h}$. *Denote with* $\eta_{\mathrm{EQ},T}$ *the local error indicators defined in* (4.12).

(i) *Reliability of* η_{EQ}. *Given any element* $T\in\mathscr{T}$ *there exists a constant* $C_T(\mathbb{H},\mathbb{C})$ *dependent on the material properties and regularity of the triangulation such that*

$$\|\sigma_\theta - \sigma_{\theta,h}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leqslant (d\lambda + 2\mu)\left(\sum_{T\in\mathscr{T}} C_T^2(\mathbb{H},\mathbb{C})\eta^2_{\mathrm{EQ},T}\right)^{1/2}. \tag{4.14}$$

(ii) *Efficiency of* η_{EQ}. *There exists a positive constant* C_{eff} *depending on the regularity of the triangulation and on the polynomial order such that*

$$\eta_{\mathrm{EQ}} - \mathrm{hot} \leqslant C_{\mathrm{eff}}\|\sigma_\theta - \sigma_{\theta,h}\|_{L^2(\Omega;\mathbb{R}^{d\times d})}. \tag{4.15}$$

For the explicit expressions of the constants C_T that enter Eq. (4.14) we refer to the material model given in Section 3. Different cases need to be distinguished according to the conditions associated with the solution u_θ of Problem 2.1 and the solution $u_{\theta,h}$ of Problem 2.3. These are summarized by the following proposition.

Proposition 4.1 (Hardening depending reliability). *Suppose* u_θ *and* $u_{\theta,h}$ *satisfy at any point* $x\in T$ *one of the following conditions*:

(i) *If elastic deformations occur in correspondence of both* u_θ *and* $u_{\theta,h}$, *then holds*

$$C_T = (2\mu)^{-1}. \tag{4.16}$$

(ii) *If inelastic deformations occur in correspondence of both* u_θ *and* $u_{\theta,h}$, *then there holds*

$$C_T = (2\mu)^{-1} + 2\theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t). \tag{4.17}$$

(iii) *If inelastic deformations occur in correspondence of either* u_θ *or* $u_{\theta,h}$, *then holds*

$$C_T = (2\mu)^{-1} + \theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t). \tag{4.18}$$

Remark 4.1. According to Proposition 4.1 (4.14) is not a full a posteriori error estimate, since the constant C_T depends on u_θ. In practice, we consider only the conditions defined by the finite element approximation $u_{\theta,h}$, with the constant given by Eq. (4.17) upon the occurrence of inelastic deformations.

4.3. Proofs of main results

The proof of Theorem 4.2 is obtained in steps. We first recall the following results and prove Proposition 4.1.

Lemma 4.1. *Let* $(u_\theta, \sigma_\theta) \in V \times \mathscr{S}$ *and* $(u_{\theta,h}, \sigma_{\theta,h}) \in V_h \times \mathscr{S}$ *be solution of Problems* 2.1 *and* 2.3, *respectively. Then there holds*

$$(2\mu + d\lambda)^{-1}\|\sigma_\theta - \sigma_{\theta,h}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \leqslant \|\sigma_\theta - \sigma_{\theta,h}\|^2_{\mathbb{C}^{-1}} := \int_\Omega (\sigma_\theta - \sigma_{\theta,h}) : \mathbb{C}^{-1}(\sigma_\theta - \sigma_{\theta,h})\mathrm{d}x$$
$$\leqslant \int_\Omega (\sigma_\theta - \sigma_{\theta,h}) : \varepsilon(u_\theta - u_{\theta,h})\mathrm{d}x. \tag{4.19}$$

Proof. The first inequality follows easily from the decomposition in spheric and deviatoric part of the stress tensor and of its expression in terms of the elasticity tensor. The second inequality follows from the same arguments of Remark 2.1 concerning the monotony of the operator $\epsilon_\theta \mapsto \sigma_\theta$. □

Lemma 4.2 (Pointwise bound on the error on the elastic strain tensor). *Denote with* $e = \mathbb{C}^{-1}\sigma$ *the elastic part of the total strain. Then there holds*

$$|e_\theta - e_{\theta,h}| \leqslant (2\mu)^{-1}|\sigma_\theta - \sigma_{\theta,h}|. \tag{4.20}$$

Proof. This follows from $\sigma = 2\mu e + \lambda\,\mathrm{tr}\,eI$. □

Lemma 4.3 (Pointwise bound on the error on the plastic strain tensor). *There exists a material dependent constant* C_H *such that there holds*

$$|p_\theta - p_{\theta,h}| \leqslant C_\mathrm{H}|\sigma_\theta - \sigma_{\theta,h}|, \tag{4.21}$$

with

(i) $C_\mathrm{H} = 2\theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t)$ *if inelastic deformations occur in correspondence of* u_θ *and* $u_{\theta,h}$;
(ii) $C_\mathrm{H} = \theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t)$ *if inelastic deformations are associated with either* u_θ *or* $u_{\theta,h}$.

Proof. (i) Let $\gamma = \theta\Delta t/(3\varrho)$. By solving $(3.7)_3$ with respect to a_θ and replacing into $(3.7)_2$, and taking the deviator of both sides, one obtains

$$\mathrm{dev}(\xi_\theta - \xi_0) = \frac{\gamma}{1+\gamma H}(|\mathrm{dev}(\sigma_\theta - k\xi_\theta)| - (\sigma_\mathrm{y} + Ha_0))\frac{\mathrm{dev}(\sigma_\theta - k\xi_\theta)}{|\mathrm{dev}(\sigma_\theta - k\xi_\theta)|}, \tag{4.22}$$

which also yields

$$|\mathrm{dev}(\sigma_\theta - k\xi_\theta)| = \frac{1+\gamma H}{\gamma}|\mathrm{dev}(\xi_\theta - \xi_0)| + (\sigma_\mathrm{y} + Ha_0). \tag{4.23}$$

Since $a_\theta \geqslant a_0$, it is

$$|\mathrm{dev}(\sigma_\theta - k\xi_\theta)| - (\sigma_\mathrm{y} + a_0 H) \geqslant 0, \tag{4.24}$$

therefore, by accounting of Eqs. (4.23) and (4.24), (4.22) reads as

$$\mathrm{dev}(\xi_\theta - \xi_0) = \frac{|\mathrm{dev}(\xi_\theta - \xi_0)|}{|\mathrm{dev}(\sigma_\theta - k\xi_\theta)|}\mathrm{dev}(\sigma_\theta - k\xi_\theta). \tag{4.25}$$

From Eqs. (4.23) and (4.25), after some rearrangements, one obtains

$$\operatorname{dev}(\xi_\theta - \xi_0) = \frac{\gamma|\operatorname{dev}(\xi_\theta - \xi_0)|\operatorname{dev}(\sigma_\theta - k\xi_0)}{(1+\gamma H + \gamma k)|\operatorname{dev}(\xi_\theta - \xi_0)| + \gamma(\sigma_{\mathrm{y}} + Ha_0)}, \tag{4.26}$$

which gives

$$|\operatorname{dev}(\xi_\theta - \xi_0)| = \frac{\gamma(|\operatorname{dev}(\sigma_\theta - k\xi_0)| - (\sigma_{\mathrm{y}} + Ha_0))}{1+\gamma H + \gamma k}. \tag{4.27}$$

By replacing then (4.27) into (4.26), one finally obtains

$$\operatorname{dev}(\xi_\theta - \xi_0) = \frac{\gamma}{1+\gamma H+\gamma k}\frac{|\operatorname{dev}(\sigma_\theta - k\xi_0)| - (\sigma_{\mathrm{y}} + Ha_0)}{|\operatorname{dev}(\sigma_\theta - k\xi_0)|}\operatorname{dev}(\sigma_\theta - k\xi_0). \tag{4.28}$$

Likewise, it is

$$\operatorname{dev}(\xi_{\theta,h} - \xi_0) = \frac{\gamma}{1+\gamma H+\gamma k}\frac{|\operatorname{dev}(\sigma_{\theta,h} - k\xi_0)| - (\sigma_{\mathrm{y}} + Ha_0)}{|\operatorname{dev}(\sigma_{\theta,h} - k\xi_0)|}\operatorname{dev}(\sigma_{\theta,h} - k\xi_0). \tag{4.29}$$

For the ease of the proof and to simplify the notation we introduce the shorthand

$$X := \operatorname{dev}(\sigma_{\theta,h} - k\xi_0), \quad Y := \operatorname{dev}(\sigma_{\theta,h} - k\xi_0),$$
$$x := \frac{\gamma}{1+\gamma H+\gamma k}\frac{|X| - (\sigma_{\mathrm{y}} + Ha_0)}{|X|}, \quad y := \frac{\gamma}{1+\gamma H+\gamma k}\frac{|Y| - (\sigma_{\mathrm{y}} + Ha_0)}{|Y|}.$$

By combining Eqs. (4.28) and (4.29), it follows:

$$\operatorname{dev}(\xi_\theta - \xi_{\theta,h}) = xX - yY = \tfrac{1}{2}(x-y)(X+Y) + \tfrac{1}{2}(x+y)(X-Y). \tag{4.30}$$

From $(3.7)_2$ and the corresponding equation of the fully discrete scheme, one can easily state that

$$|\xi_\theta - \xi_{\theta,h}| = |\operatorname{dev}(\xi_\theta - \xi_{\theta,h})|, \tag{4.31}$$

so that applications of the triangular inequality in (4.30) produce

$$|\xi_\theta - \xi_{\theta,h}| \leqslant \tfrac{1}{2}|x-y|(|X|+|Y|) + \tfrac{1}{2}|x+y||X-Y|. \tag{4.32}$$

Also, it is

$$x + y = \frac{\gamma}{1+\gamma H+\gamma k}\left(\frac{|X| - (\sigma_{\mathrm{y}} + Ha_0)}{|X|} + \frac{|Y| - (\sigma_{\mathrm{y}} + Ha_0)}{|Y|}\right) \leqslant \frac{2\gamma}{1+\gamma H+\gamma k}. \tag{4.33}$$

After some rearrangements, one obtains

$$|x-y|(|X|+|Y|) = \frac{\gamma}{1+\gamma H+\gamma k}\frac{\sigma_{\mathrm{y}} + Ha_0}{|X||Y|}(|X|+|Y|)||X|-|Y|| \leqslant \frac{2\gamma}{1+\gamma H+\gamma k}|X-Y|. \tag{4.34}$$

By accounting of the expression of γ (4.32) can be recast as

$$|\xi_\theta - \xi_{\theta,h}| \leqslant \frac{2\gamma}{1+\gamma H+\gamma k}|\operatorname{dev}(\sigma_\theta - \sigma_{\theta,h})| \leqslant \frac{2\theta\Delta t}{3\varrho + H\theta\Delta t + k\theta\Delta t}|\sigma_\theta - \sigma_{\theta,h}|. \tag{4.35}$$

Since $p_\theta - p_{\theta,h} = \xi_\theta - \xi_{\theta,h}$, one proves (4.21) for $C_{\mathrm{H}} = 2\theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t)$.

(ii) If, for instance, $u_{\theta,h}$ does not produce inelastic deformations, it is

$$p_{\theta,h} = p_0, \quad \xi_{\theta,h} = \xi_0, \quad a_{\theta,h} = a_0, \quad |\operatorname{dev}(\sigma_{\theta,h} - k\xi_0)| - (\sigma_{\mathrm{y}} + Ha_0) \leqslant 0. \tag{4.36}$$

On the other hand, since u_θ produces inelastic deformations, Eq. (4.28) holds, and accounting for $(4.36)_4$ it follows:

$$
\begin{aligned}
|\mathrm{dev}(\xi_\theta - \xi_0)| &= \frac{\gamma}{1+\gamma H+\gamma k}\Big| |\mathrm{dev}(\sigma_\theta - k\xi_0)| - (\sigma_{\mathrm{y}} + Ha_0)\Big| \\
&\leqslant \frac{\gamma}{1+\gamma H+\gamma k}\Big| |\mathrm{dev}(\sigma_\theta - k\xi_0)| - |\mathrm{dev}(\sigma_{\theta,h} - k\xi_0)|\Big| \leqslant \frac{\gamma}{1+\gamma H+\gamma k}|\sigma_\theta - \sigma_{\theta,h}|.
\end{aligned} \tag{4.37}
$$

As a result, from $(4.36)_2$, $(3.7)_2$ and the expression of γ, one finds

$$
|\xi_\theta - \xi_{\theta,h}| = |\xi_\theta - \xi_{\theta,0}| = |\mathrm{dev}(\xi_\theta - \xi_{\theta,0})| \leqslant \frac{\theta\Delta t}{3\varrho + H\theta\Delta t + k\theta\Delta t}|\sigma_\theta - \sigma_{\theta,h}|, \tag{4.38}
$$

proving (4.21) with $C_{\mathrm{H}} = \theta\Delta t/(3\varrho + H\theta\Delta t + k\theta\Delta t)$. □

Proof of Proposition 4.1. From the additivity of the total strain,

$$
\varepsilon(u_\theta - u_{\theta,h}) = e_\theta - e_{\theta,h} + p_\theta - p_{\theta,h}, \tag{4.39}
$$

combining the results of Lemmas 4.2 and 4.3, and using the triangular inequality, one obtains the following pointwise bound for the norm of the total strain tensor

$$
|\varepsilon(u_\theta - u_{\theta,h})| \leqslant |e_\theta - e_{\theta,h}| + |p_\theta - p_{\theta,h}| \leqslant ((2\mu)^{-1} + C_{\mathrm{H}})|\sigma_\theta - \sigma_{\theta,h}| \tag{4.40}
$$

with C_{H} varying according to the cases of Lemma 4.3. □

From (4.40), and assuming C_{H} constant over T, one concludes that given any $T \in \mathscr{T}$ there holds

$$
\|\varepsilon(u_\theta - u_{\theta,h})\|_{L^2(T;\mathbb{R}^{d\times d})} \leqslant C_T(\mathbb{H},\mathbb{C})\|\sigma_\theta - \sigma_{\theta,h}\|_{L^2(T;\mathbb{R}^{d\times d})}, \tag{4.41}
$$

where we have let $C_T(\mathbb{H},\mathbb{C}) := (2\mu)^{-1} + C_{\mathrm{H}}$.

Proof of Theorem 4.2. (i) Reliability of η_{EQ}. The Galerkin orthogonality and Lemma 4.1 yield

$$
\frac{1}{d\lambda + 2\mu}\|\sigma_\theta - \sigma_{\theta,h}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \leqslant \int_\Omega (\sigma_\theta - \sigma_{\theta,h}) : \varepsilon(u_\theta - u_{\theta,h})\mathrm{d}x = \langle R_h, u_\theta - u_{\theta,h}\rangle. \tag{4.42}
$$

Recalling the localization of the residual at any $T \in \mathscr{T}$ given by (4.9) and Theorem 4.1 it follows that

$$
\langle R_h, u_\theta - u_{\theta,h}\rangle = \sum_{T\in\mathscr{T}} \ell_T(u_\theta - u_{\theta,h}) = \sum_{T\in\mathscr{T}} \int_T \varepsilon(\phi_T) : \varepsilon(u_\theta - u_{\theta,h})\mathrm{d}x. \tag{4.43}
$$

Using Cauchy–Schwarz inequality and (4.41), one obtains

$$
\begin{aligned}
\int_T \varepsilon(\phi_T) : \varepsilon(u_\theta - u_{\theta,h})\mathrm{d}x &\leqslant \|\varepsilon(\phi_T)\|_{L^2(T;\mathbb{R}^{d\times d})}\|\varepsilon(u_\theta - u_{\theta,h})\|_{L^2(T;\mathbb{R}^{d\times d})} \\
&\leqslant \|\varepsilon(\phi_T)\|_{L^2(T;\mathbb{R}^{d\times d})} C_T(\mathbb{H},\mathbb{C})\|\sigma_\theta - \sigma_{\theta,h}\|_{L^2(T;\mathbb{R}^{d\times d})}.
\end{aligned} \tag{4.44}
$$

Summing over each element and the discrete form of the Cauchy–Schwarz inequality yield

$$
\begin{aligned}
&\sum_{T\in\mathscr{T}} \int_T \varepsilon(\phi_T) : \varepsilon(u_\theta - u_{\theta,h})\mathrm{d}x \\
&\quad \leqslant \left(\sum_{T\in\mathscr{T}} C_T^2(\mathbb{H},\mathbb{C})\|\varepsilon(\phi_T)\|^2_{L^2(T;\mathbb{R}^{d\times d})}\right)^{1/2} \left(\sum_{T\in\mathscr{T}} \|\sigma_\theta - \sigma_{\theta,h}\|^2_{L^2(T;\mathbb{R}^{d\times d})}\right)^{1/2}.
\end{aligned} \tag{4.45}
$$

By combining Eqs. (4.42), (4.43), and (4.45), one finally obtains (4.14).

(ii) Efficiency of η_{EQ}. By accounting of (4.10) and corresponding modification if $E \in \mathscr{E}_{\mathrm{N}} \cup \mathscr{E}_{\mathrm{D}}$, rearrange the terms in (4.8) in the form

$$\ell_T^{\mathrm{EQ}}(v) = \int_T r_{h,T}\cdot v\,\mathrm{d}x + \sum_{E\in\mathscr{E}(T)}\int_{\partial E}(g_E|_T - \langle\sigma_h v_E\rangle)\cdot v\,\mathrm{d}s + \frac{1}{2}\sum_{E\in\mathscr{E}(T)}\int_E J_{h,E}\cdot v\,\mathrm{d}s, \tag{4.46}$$

with

$$\langle\sigma_h v_E\rangle = \begin{cases}\frac{1}{2}(\sigma_h|_T + \sigma_h|_{T'})v_E & \text{on } \mathscr{E}(T)\cap\mathscr{E}(T'),\\ \sigma_h|_T v_E & \text{on } \mathscr{E}(T)\cap\mathscr{E}_{\mathrm{D}},\\ g & \text{on } \mathscr{E}(T)\cap\mathscr{E}_{\mathrm{N}}.\end{cases}$$

For the edge $E\in\mathscr{E}\setminus\Gamma_{\mathrm{N}}$ introduce the L^2-orthogonal projection $\Pi_{E,k}$ of $L^2(E;\mathbb{R}^d)$ onto $P_k(E;\mathbb{R}^d)$. For the equilibration condition $(4.8)_2$, and with $g_E|_T\in P_k(E;\mathbb{R}^d)$, there holds [26, Thm. 4.41], [1, Thm. 3]

$$h_T\sum_{E\in\mathscr{E}(T)}\|g_T - \Pi_{E,k}\langle\sigma_h v_E\rangle\|^2_{L^2(E;\mathbb{R}^d)} \leqslant C\left(h_T^2\sum_{T'\subseteq\omega(T)}\|r_{h,T'}\|^2_{L^2(T';\mathbb{R}^d)} + h_T\sum_{E\in\overset{\circ}{\mathscr{E}}(\omega(T))}\|J_{h,E}\|^2_{L^2(E;\mathbb{R}^d)}\right), \tag{4.47}$$

with C a positive constant depending on the polynomial degree k and not on the mesh size. After introducing the mean value of v as $\bar{v} = 1/|T|\int_T v\,\mathrm{d}x$, from Eqs. (4.11) and (4.46) it follows:

$$\begin{aligned}\int_T \varepsilon(\phi_T):\varepsilon(v)\mathrm{d}x &= \int_T \varepsilon(\phi_T):\varepsilon(v-\bar{v})\mathrm{d}x = \ell_T^{\mathrm{EQ}}(v-\bar{v})\\ &= \int_T r_{h,T}\cdot(v-\bar{v})\mathrm{d}x + \frac{1}{2}\sum_{E\in\mathscr{E}(T)}\int_E J_{h,E}\cdot(v-\bar{v})\mathrm{d}s\\ &\quad + \sum_{E\in\mathscr{E}(T)}\int_E(g_E - \langle\sigma_h v_E\rangle)\cdot(v-\bar{v})\mathrm{d}s.\end{aligned} \tag{4.48}$$

For $v=\phi_T$ in (4.48), the Cauchy inequality first, and the Poincaré inequality then, give

$$\begin{aligned}\eta^2_{\mathrm{EQ},T} \leqslant C\|D\phi_T\|_{L^2(T;\mathbb{R}^{d\times d})}\Bigg(&\|h_T r_{h,T}\|_{L^2(T;\mathbb{R}^d)} + \sum_{E\in\mathscr{E}(T)}\|h_E^{1/2}J_{h,E}\|_{L^2(E;\mathbb{R}^d)}\\ &+ \sum_{E\in\mathscr{E}(T)}\|h_E^{1/2}(g_E|_T - \Pi_{E,k}\langle\sigma_h v_E\rangle)\|_{L^2(E;\mathbb{R}^d)} + \sum_{E\in\mathscr{E}(T)}\|h_E^{1/2}(\Pi_{E,k}\langle\sigma_h v_E\rangle - \langle\sigma_h v_E\rangle)\|_{L^2(E;\mathbb{R}^d)}\Bigg).\end{aligned}$$

The Korn inequality $\|D\phi_T\|_{L^2(T;\mathbb{R}^{d\times d})}\leqslant C\|\varepsilon(\phi_T)\|_{L^2(T;\mathbb{R}^{d\times d})}$, the definition (4.12) of $\eta_{\mathrm{EQ},T}$, summing over $T'\in\omega(T)$, and the estimate (4.47) finally yield the following local efficiency estimate

$$\eta_{\mathrm{EQ},T}\leqslant C\left(\|hr_h\|_{L^2(\omega(T);\mathbb{R}^d)} + \sum_{E\in\overset{\circ}{\mathscr{E}}(\omega(T))}\|h_E^{1/2}J_{h,E}\|_{L^2(E;\mathbb{R}^d)} + \sum_{E\in\overset{\circ}{\mathscr{E}}(\omega(T))}\|h_E^{1/2}(\Pi_{E,k}\langle\sigma_h v_E\rangle - \langle\sigma_h v_E\rangle)\|_{L^2(E;\mathbb{R}^d)}\right), \tag{4.49}$$

with C depending on the shape of Ω and on the polynomial degree k.

Following the technique with inverse estimates due to Verfürth [41,42], denote by $\tilde{f}$ and $\tilde{g}$ a finite element approximation of f and g, for instance piecewise constant. Define $\widetilde{J}_E = \tilde{g} - \sigma_h v$ if $E\in\mathscr{E}_{\mathrm{N}}$, otherwise $\widetilde{J}_E = J_{h,E}$, then there holds

$$\begin{aligned}&\|h_T r_{h,T}\|_{L^2(T;\mathbb{R}^d)} \leqslant C(\|\sigma-\sigma_h\|_{L^2(\omega(T);\mathbb{R}^{d\times d})} + \|h(f-\tilde{f})\|_{L^2(\omega(T);\mathbb{R}^d)}),\\ &\|h_E^{1/2}J_{h,E}\|_{L^2(E;\mathbb{R}^d)} \leqslant C(\|\sigma-\sigma_h\|_{L^2(\omega(E);\mathbb{R}^{d\times d})} + \|h(f-\tilde{f})\|_{L^2(\omega(E);\mathbb{R}^d)} + \|h_E^{1/2}(J_{h,E}-\tilde{J}_E)\|_{L^2(E;\mathbb{R}^d)}),\end{aligned} \tag{4.50}$$

with C a positive constant depending on the shape regularity of the triangulation. Hence, (4.50) together with (4.49), and the finite overlap between the patches of elements, establish finally (4.15). □

5. Adaptive algorithm and numerical examples

This section is devoted to numerical experiments on a posteriori error control and h-finite element adaptive algorithms for a single time step of the elastoplastic and viscoplastic evolution problem. An h-finite element adaptive algorithm consists of recursive loops of the form [9].

SOLVE → ESTIMATE → MARK → REFINE

All the discretizations are generated by Algorithm 5.1 where for the step ESTIMATE we use η_{R} and η_{EQ}, whereas the step MARK is defined by the max-refinement rule that marks a subset $\mathscr{M} \subseteq \mathscr{E}$ according to (5.1). We consider $\Theta = 0$ for uniform meshes (as all elements in step (e) are marked) and $\Theta = 1/2$ for adaptive mesh-refining (related strategies and a different choice for $0 < \Theta < 1$ are disputable). Notice that the algorithm is the same as in linear elasticity, for the estimates η_{EQ} and η_{R} are related only to the norm of the residual in the equilibrium equations.

Algorithm 5.1

(a) Start with a coarse mesh $\mathscr{T}_0$, set $k = 0$.
(b) Solve the discrete problem with respect to the actual mesh $\mathscr{T}_k$ for N degrees of freedom.
(c) Compute $\eta_T = \eta_{\mathrm{EQ},T}$ (resp. $\eta_T = \eta_{\mathrm{R},T}$) for all $T \in \mathscr{T}_k$.
(d) Compute $\eta_{\mathrm{N}} := \left(\sum_{T\in\mathscr{T}} \eta_T^2\right)^{1/2}$ as estimate for $e_{\mathrm{N}} := \|\sigma_\theta - \sigma_{\theta,h}\|_{L^2(\Omega)}$.
(e) Mark the element T for (*red*) refinement provided

$$\Theta \max_{T' \in \mathscr{T}_k} \eta_{T'} \leqslant \eta_T. \tag{5.1}$$

(f) Mark further elements (within a *red–green–blue* refinement) to avoid hanging nodes. Define the resulting mesh as the actual mesh $\mathscr{T}_{k+1}$, update k and go to (b).

Details on the so-called *red–green–blue* refinement strategies may be found in [41].

In the following three numerical examples, all involving linear kinematic hardening, a backward Euler time scheme has been employed for the time discretization, whereas the space discretization has been obtained with P_2 finite elements in order to prevent locking [43]. For the evaluation of the equilibrated error estimates, the flux splitting detailed in [3] with *first-order* equilibration has been assumed. Computable approximations to the solution of the local Neumann problems (4.11) have been obtained using a P_4-finite element method which delivers reliable and efficient computed estimates as well, provided that the given data are smooth.

In the first example, with known analytical solution, we analyse the behaviour of the exact error $e := \|\sigma_\theta - \sigma_{\theta,h}\|_{L^2}$ and of the bounds $C_{\mathrm{rel}}\eta_{\mathrm{R}}$, $C_{\mathrm{rel}}\eta_{\mathrm{EQ}}$ with the hardening modulus k and the number of degree of freedom. In the next two examples, for the presence of a singularity in the stress field and motivated by the efficiency estimate, we compare uniform and adaptive refinements based on η_{EQ} and η_{R}.

5.1. Elastoplastic ring with known solution

Consider the ring depicted in Fig. 2 with no volume force ($f = 0$) and radially applied surface forces $g_1(r,\phi,t) = te_r$ and $g_2(r,\phi,t) = -t/4e_r$, where $e_r = (\cos\phi, \sin\phi)$.

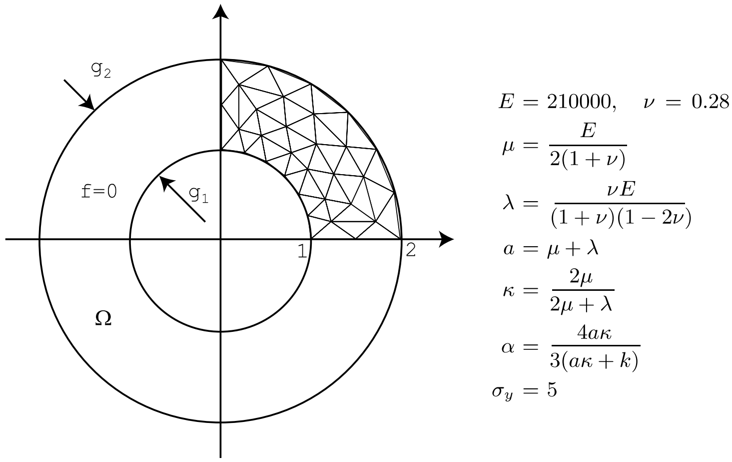

Fig. 2. Geometric model for the ring with elastoplastic and kinematic hardening material. Initial mesh $\mathscr{T}_0$: 48 P_2-elements.

The analytical solution for the body centred at the origin with no rotation reads

$$u(r,\phi,t) = u_r(r,t)e_r, \tag{5.2}$$

$$\sigma(r,\phi,t) = \sigma_r(r,t)e_r \otimes e_r + \sigma_\phi(r,t)e_\phi \otimes e_\phi, \tag{5.3}$$

$$p(r,\phi,t) = p_r(r,t)(e_r \otimes e_r - e_\phi \otimes e_\phi) \tag{5.4}$$

(see [12] for details) with $e_\phi = (-\sin\phi, \cos\phi)$ and

$$\begin{aligned}
&u_r(r,t) = t/(2\mu r) - (2/3)\kappa(r + 4a/(\mu r))I(1) - 2\kappa r I(r),\\
&\sigma_r(r,t) = -t/r^2 - (2/3)a\kappa(1 - 4/r^2)I(1) - 2a\kappa I(r),\\
&\sigma_\phi(r,t) = \partial(r\sigma_r)/\partial r,\\
&p_r(r,t) = -\sigma_y/(\sqrt{2}(a\kappa + k))(R^2/r^2 - 1)_+,\\
&I(r) = -\sigma_y/(\sqrt{2}(a\kappa + k))\big(1/2(R^2/r^2 - 1)_+ - (\ln(R/r))_+\big).
\end{aligned}$$

The radius $R(t)$ of the circular plastic boundary is the positive root of

$$\alpha \ln R^2 = (\alpha - 1)R^2 - \alpha + t\sqrt{2}/\sigma_y.$$

A uniform time-step size 0.1 over the time interval $(0, 4.0)$ has been used and according to symmetry, only a quarter of the domain has been discretized with symmetric boundary conditions and the initial coarse mesh $\mathscr{T}_0$ of Fig. 2. Within each refinement step (f) of Algorithm 5.1, new nodes on the boundary are projected onto the curved boundary.

For only purpose of comparison we consider here the worst scenario given by the occurrence of inelastic deformations in correspondence of both u_{n+1} and $u_{n+1,h}$, and C_T from (4.17) constant over each element. In presence of viscosity from (4.14) follows:

$$C_{\mathrm{rel}} = (d\lambda + 2\mu)C_T = (d\lambda + 2\mu)((2\mu)^{-1} + 2/(3\rho/\Delta t + k)) \tag{5.5}$$

with $d = 2$. The values of C_{rel} for different values of k and $\rho/\Delta t$ are reported in Table 1. Although the presence of viscosity reduces the value of C_{rel} for small values of k, its value still remains prohibitive for providing useful upper bounds. On contrary, the influence of ρ is not relevant for values of k close to the elastic modulus.

Table 1
Variation of the reliability constant C_{rel} from Eq. (5.5) with the hardening modulus k and the ratio $\rho/\Delta t$

k	C_{rel}				
	$\rho/\Delta t=0$	$\rho/\Delta t=1$	$\rho/\Delta t=10$	$\rho/\Delta t=100$	$\rho/\Delta t=1000$
0.1	7,457,389	240,563	24,778	2487	251
1	745,740	186,437	24,058	2480	251
10	74,576	57,367	18,646	2409	250
100	7460	7242	5739	1867	243
1000	748	746	726	576	189
10,000	77	77	77	75	60
28,000	29	29	29	29	26
50,000	17	17	17	17	16
100,000	9.7	9.7	9.7	9.7	9.5
150,000	7.2	7.2	7.2	7.2	7.1
200,000	6.0	6.0	6.0	6.0	5.9

Notice that for useful values of C_{rel}, the influence of the ratio $\rho/\Delta t$ is small compared to k. The elastic material constants are $E=210{,}000$, $v=0.28$.

For the model problem under consideration, with no viscosity, Table 2 reports the error on the stress for different values of the hardening modulus k, along with the value of the estimates η_{EQ} and η_{R}, the reliability constant C_{rel} as from (5.5) for $\varrho\to 0$, and the effectivity index

$$\zeta=(d\lambda+2\mu)((2\mu)^{-1}+2/k)\eta/\|\sigma-\sigma_h\|_{L^2}. \tag{5.6}$$

For $\eta=\eta_{\mathrm{EQ}}$ and $\eta=\eta_{\mathrm{R}}$, (5.6) gives ζ_{EQ} and ζ_{R}, respectively.

The error bound $C_{\mathrm{rel}}\eta$ crucially depends on the factor C_{rel}. Apart from the critical dependence on the parameter v, present also in elasticity, there is a by far problematic dependence on k. The value of this variable is critical in the comparison with the elasticity modulus. It is seen that the effectivity index results in an acceptable value only for k close to E. For small values of k compared with the elasticity modulus E the effectivity index is very high and for $k\to 0$, $C_{\mathrm{rel}}\to\infty$. This corresponds to the case of perfect plasticity, which is therefore ruled out in our considerations. Furthermore, it is interesting to notice that when we compare only η with the exact error e, the equilibrated estimate η_{EQ} is closer to e than η_{R}.

Table 2
Variation of the effectivity index and of the value of the reliability constant with the hardening parameter k

k	$\|\sigma-\sigma_h\|_2$	η_{EQ}	η_{R}	C_{rel}	ζ_{EQ}	ζ_{R}
150,000	0.000893	0.001449	0.004084	7.2	11.8	33
100,000	0.001001	0.001608	0.004747	9.7	15.6	46
80,000	0.001055	0.001683	0.005121	12	18.59	56
50,000	0.001141	0.001791	0.005807	17	27	87
28,000	0.001188	0.001853	0.006302	29	45	153
10,000	0.001194	0.001869	0.006910	77	120	444
1000	0.001179	0.001848	0.007177	748	1172	4550
100	0.001177	0.001846	0.007212	7460	11,695	45,703

It is noticed that the value of C_{rel}, ζ_{EQ}, and ζ_{R} depend crucially on k, with high overestimation for values of k small compared to the elastic modulus. The values reported in the table refers to the analysis carried out in the last time step on a mesh with 49,666 degree of freedom.

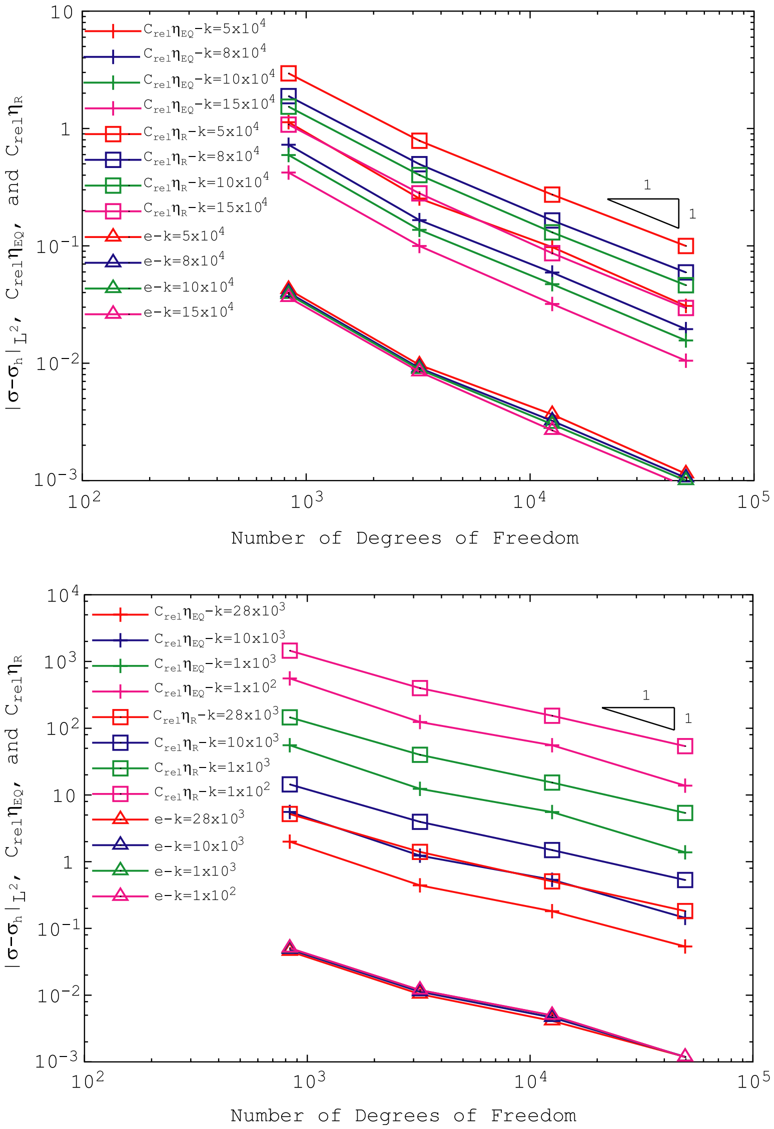

Fig. 3. Convergence history of the error, of the bound $C_{\rm rel}\eta_{\rm EQ}$, and $C_{\rm rel}\eta_{\rm R}$ as a function of the degrees of freedom and different values of the hardening modulus k. In the top picture the behaviour of $C_{\rm rel}\eta_{\rm EQ}$ and $C_{\rm rel}\eta_{\rm R}$ is compared to the error for values of $k = 1.5 \times 10^5$, 10^5, 8×10^4, 5×10^4 noting that both the bounds are closer to the error for $k = 1.5 \times 10^5$ whereas $C_{\rm rel}\eta_{\rm EQ}$ provides sharper bound. In the bottom picture, reduced values of k are considered, $k = 2.8 \times 10^4$, 10^4, 10^3, 10^2, with both $C_{\rm rel}\eta_{\rm R}$ and $C_{\rm rel}\eta_{\rm R}$ hugely overestimating the error.

The behaviour of the exact error e, of the bounds $C_{\rm rel}\eta_{\rm EQ}$, and $C_{\rm rel}\eta_{\rm R}$ with the number of degrees of freedom and the hardening parameter k is finally plotted in Fig. 3. A behaviour similar to the one described in Table 2 can be noted with the hardening modulus. Given the regularity of the solution, both the bounds $C_{\rm rel}\eta_{\rm EQ}$ and $C_{\rm rel}\eta_{\rm R}$ perform a convergence rate with respect to the number of degrees of freedom of about 1 for uniform refinement. This corresponds to the convergence rate of 2 with respect to a (uniform) mesh size h, consistently with the use of P_2 finite elements and to the estimate in energy norm.

5.2. L-shape domain

The L-shape domain $\Omega = (0,1)^2 \backslash (0,0.5)^2$ with vanishing volume force f and boundary conditions depicted in Fig. 4 is analysed. Only one time step, equal to the time interval $(0, 0.1)$, is considered for the time discretization

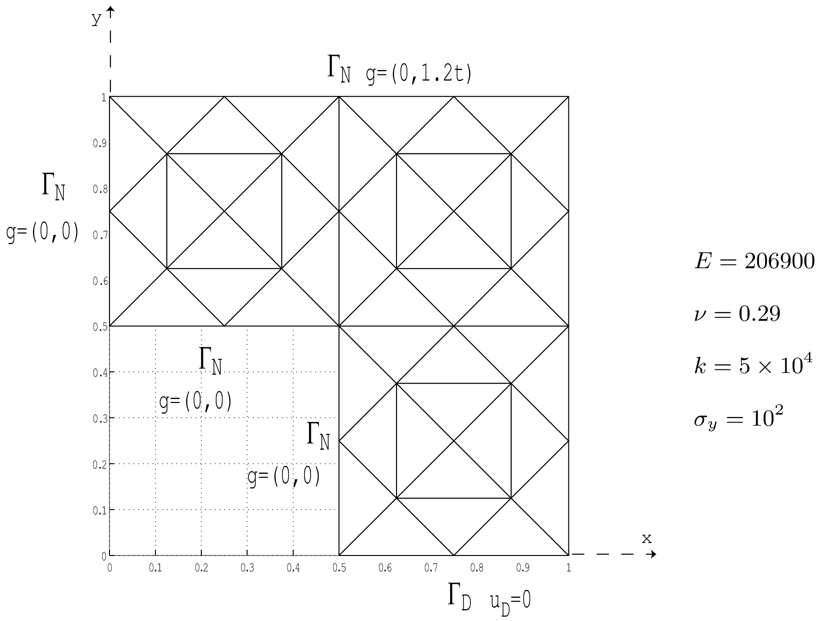

Fig. 4. Geometric model for the L-shape domain with Dirichlet boundary condition $u = 0$ on Γ_D and Neumann boundary conditions $g = (0, 1.2t)$ on Γ_N at $y = 1$ for $t \in (0,0.1)$. Initial mesh $\mathcal{T}_0$: 48 P_2-elements.

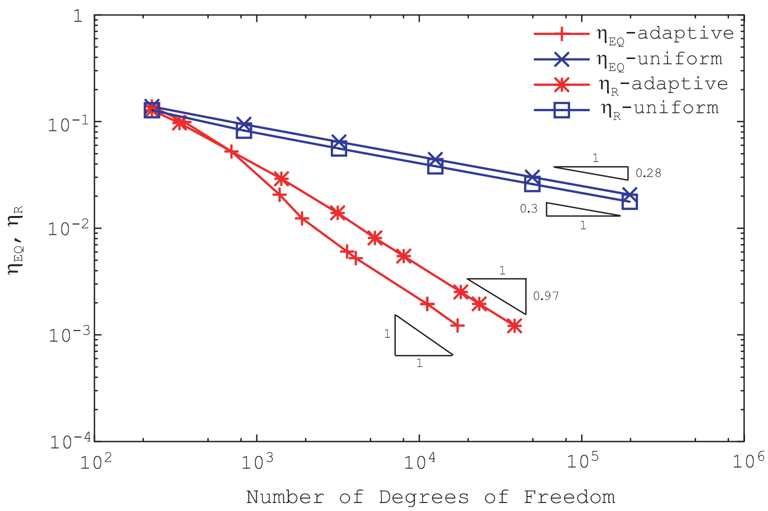

Fig. 5. Convergence rate of η_{EQ} and η_R for uniform and adaptive finite element refinements based on η_{EQ} and η_R for the L-shape domain. Both estimates η_{EQ} and η_R perform a convergence rate with respect to the number of degrees of freedom of about 1 for adaptive refinement in contrast to 0.33 for uniform refinement. This corresponds to the convergence rate of 2 and 0.66, respectively, with respect to a (uniform) mesh size h.

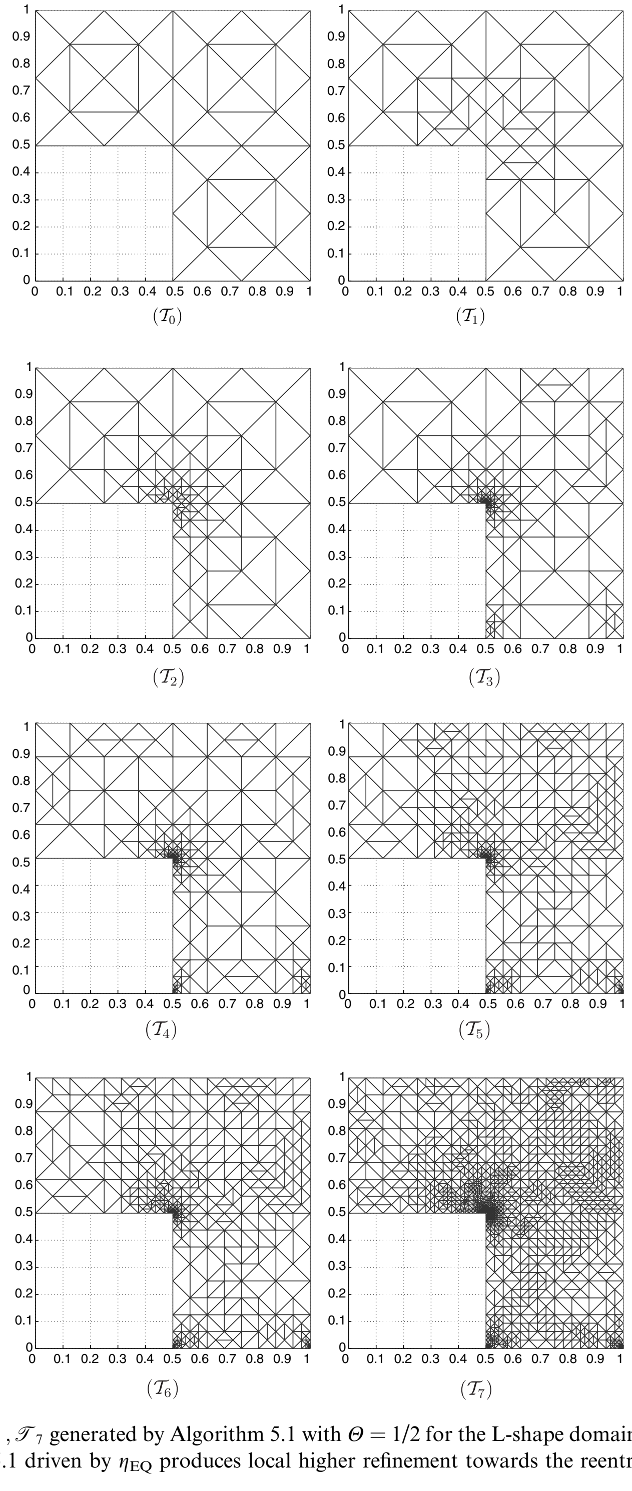

Fig. 6. Adapted meshes $\mathscr{T}_0, \ldots, \mathscr{T}_7$ generated by Algorithm 5.1 with $\Theta = 1/2$ for the L-shape domain based on a refinement with η_{EQ}. The mesh refining Algorithm 5.1 driven by η_{EQ} produces local higher refinement towards the reentrant corner.

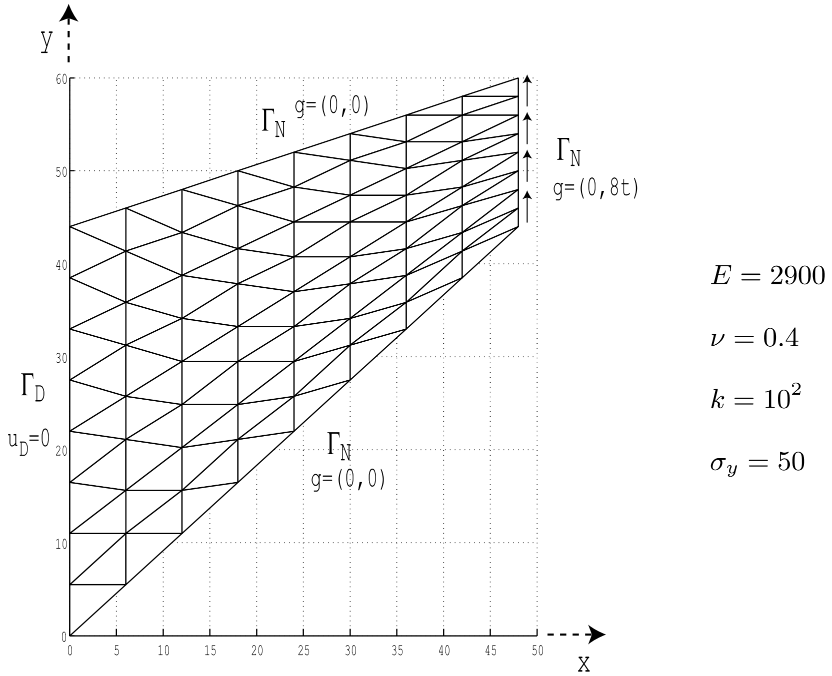

Fig. 7. Geometric model for the Cook's membrane with $u = 0$ on Γ_D and applied load $g = (0, 8t)$ on the boundary Γ_N at $x = 48$ for $t \in (0, 1.0)$. Initial mesh $\mathcal{T}_0$: 128 P_2 triangular finite elements.

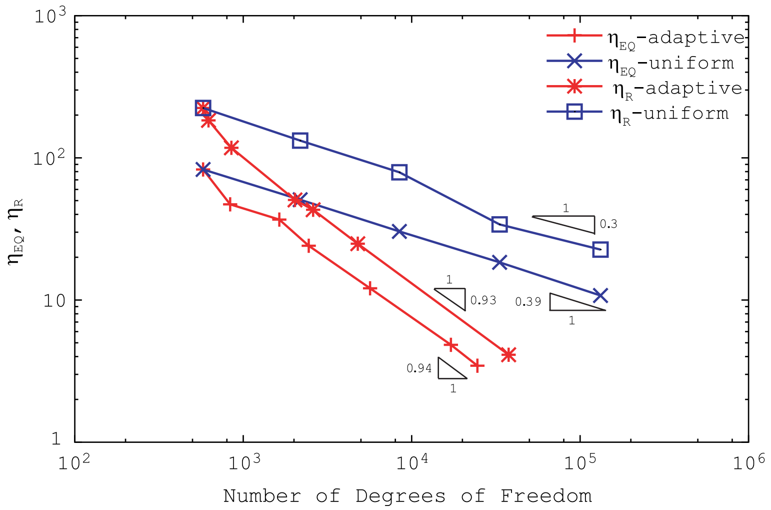

Fig. 8. Convergence rate of η_{EQ} and η_R for uniform and adaptive finite element refinements based on η_{EQ} and η_R for the Cook's membrane. Both estimates η_{EQ} and η_R perform for adaptive refinement a convergence rate of about 1 corresponding to the convergence rate 2 with respect to a (uniform) mesh size h.

of the evolution problem with the backward Euler scheme, whereas the coarse mesh $\mathcal{T}_0$ with P_2-triangular elements shown in Fig. 4 realises the space discretization in the first step of Algorithm 5.1 with $\Theta = 1/2$.

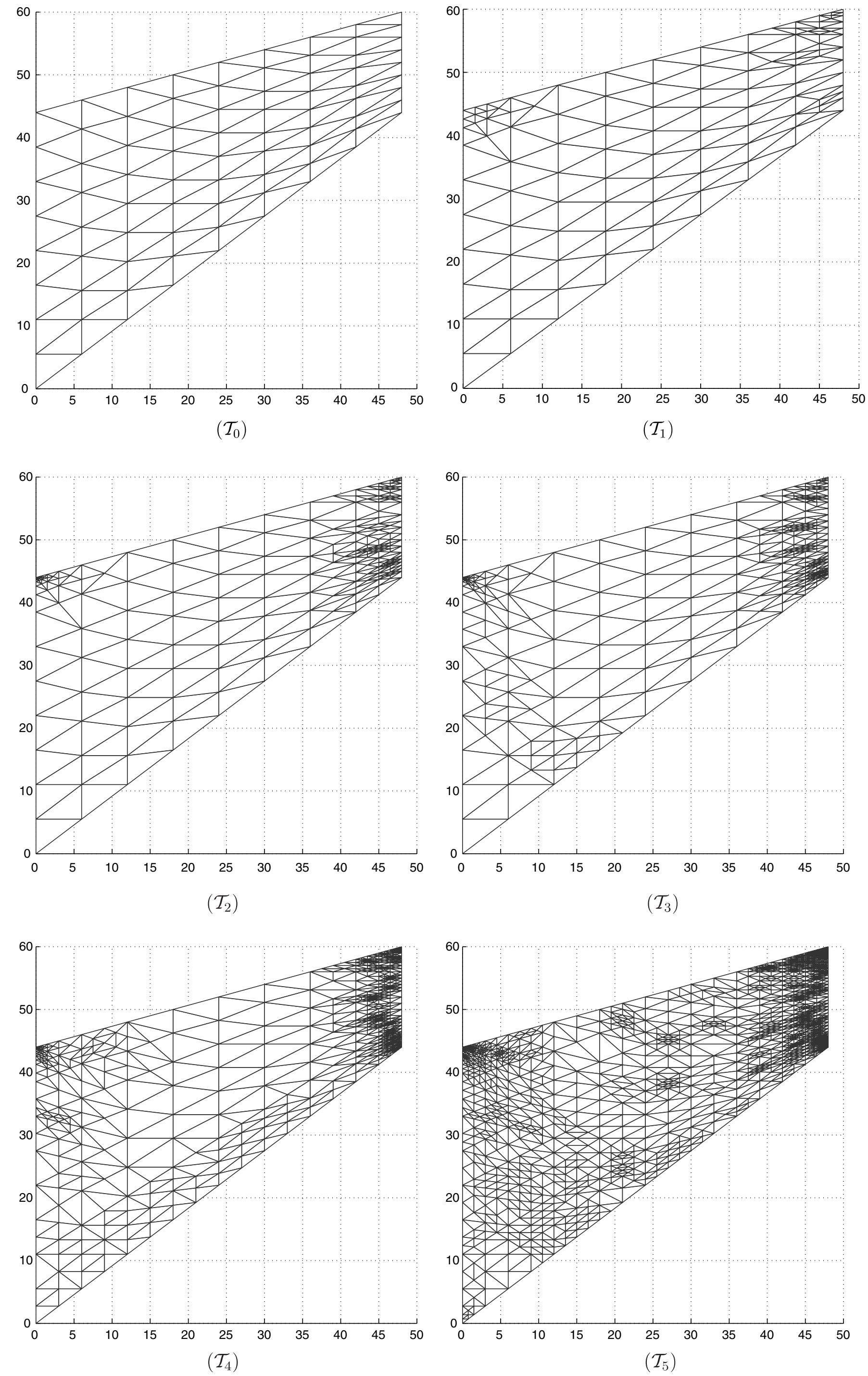

Fig. 9. Adapted triangulations $\mathscr{T}_0, \ldots, \mathscr{T}_5$ for the Cook's membrane generated with the Algorithm 5.1 with $\Theta = 1/2$, and based on a refinement with η_{EQ}. Notice a local higher refinement towards the upper left corner where change of boundary conditions occur and at the right end where point loads are applied.

For elastic behaviour, the analytical solution presents a typical corner singularity in the stress field at the reentrant corner. In the elastoplastic case the exact solution is unknown and we suspect that it might show a similar singularity at the origin. Hence, adaptive algorithms should lead to a better convergence. This is shown in Fig. 5 where convergence rates of η_{R} and η_{EQ} for uniform and adaptive refinement based on η_{R} and η_{EQ}, respectively, are compared. One can notice that the uniform mesh-refinement converges only sub-optimally due to the singularity, as expected, while adaptive mesh-refinement based on both η_{EQ} and η_{R} recovers optimal convergence rates, also on occurrence of plastic deformations. Finally, Fig. 6 depicts the triangulations generated with Algorithm 5.1 showing a local higher refinements towards the reentrant corner.

5.3. Cook's membrane

The Cook's membrane depicted in Fig. 7 is clamped at one end and subjected to a shear load $g = (0, 8.0t)$ along the opposite end (and vanishing volume force $f = 0$). Also in this example only one time step equal to the time interval $(0, 1.0)$ is considered for the time discretization of the evolution problem with the backward Euler scheme, whereas the coarse mesh $\mathscr{T}_0$ with P_2-triangular elements shown in Fig. 7 is used for the space discretization in the first step of Algorithm 5.1.

This problem as well is meant to serve as test for the performance of adaptive finite element methods for incremental plasticity. The proposed Algorithm 5.1 leads to a slightly better order of experimental convergence depicted in Fig. 8 with adaptive finite element refinement based on both η_{EQ} and η_{R}. The triangulations generated by Algorithm 5.1 based on η_{EQ}, reported in Fig. 9, show local mesh refinement towards the upper left corner where a change of the type of boundary conditions causes a singularity, and at the right end where point loads are applied.

6. Conclusions

This paper establishes reliability and efficiency of equilibrated a posteriori error estimates for L^2 stress error control of conforming displacement finite element approximations. For the explicit bounds, explicit expressions of the reliability constant are developed in terms of the hardening parameter for a quite general class of inelastic material models. Theoretical and numerical considerations illustrated the crucial dependence of the reliability constant on the hardening parameter leading to a deterioration of the guaranteed upper bound when k approaches zero. This corresponds to perfect plasticity where the estimate is *not* theoretically justified. Furthermore, motivated by the efficiency estimate, using directly η_{EQ} and/or η_{R} in the adaptive mesh-refining Algorithm 5.1 one improves the quality of the spatial discretization in case of non-optimal quasiuniform meshes. The related estimator η_{EQ} and also η_{R} showed a significantly improved experimental convergence rate for the examined model problems.

Acknowledgements

The authors C.C. and A.O. acknowledge support by the German Research Foundation (DFG) at the Research Centre "Mathematics for key technologies" under Project C13. The work of C.C. was also supported by the Austrian Science Fund (FWF) under Project P15274 and P16461 whereas R.K. was supported by the DFG-Graduiertenkolleg 357 "Effiziente Algorithmen und Mehrskalenmethoden".

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